

PERFECT AND SEMIPERFECT RESTRICTED ENVELOPING ALGEBRAS

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ABSTRACT. For a restricted Lie algebra L , the conditions under which its restricted enveloping algebra $u(L)$ is semiperfect are investigated. Moreover, it is proved that $u(L)$ is left (or right) perfect if and only if L is finite-dimensional.

1. INTRODUCTION

Let R be a ring with unity and denote by $\mathcal{J}(R)$ the Jacobson radical of R . We recall that R is said to be *semiperfect* if $R/\mathcal{J}(R)$ is Artinian and idempotents of $R/\mathcal{J}(R)$ can be lifted to R . Semiperfect rings, introduced by H. Bass in [1], turn out to be a significant class of rings from the viewpoint of homological algebra and representation theory, since they are precisely the rings R for which all finitely generated left or right R -modules have a projective cover (see e.g. [11], Chapter 8, §24). Clearly, one sided Artinian rings and local rings are semiperfect.

Recall that R is called *left perfect* if all left R -modules have projective covers. Right perfect rings are defined in an analogous way. The pioneering work on perfect rings was carried out by H. Bass in 1960 and most of the main characterizations of these rings are contained in his celebrated paper [1]. In particular, it follows from Bass' results that the following conditions are equivalent: R is left perfect; every flat left R -module is projective; $R/\mathcal{J}(R)$ is Artinian and for every sequence $\{a_i\}$ in $\mathcal{J}(R)$ there exists an integer n such that $a_1 a_2 \cdots a_n = 0$; R satisfies the descending chain condition on principal right ideals. It should be mentioned that, while semiperfectness is a left-right symmetric property, there exist rings which are perfect on one side but not on the other (see [1], Example 5 on page 476). However, right and left perfectness are clearly equivalent conditions provided R has a nontrivial involution. For instance, this is the case when R is a group algebra or an (ordinary or restricted) enveloping algebra.

Left perfect group rings were characterized by G. Renault in [17] and, independently, by S. M. Woods in [23]. It turns out that, for a group G and a field \mathbb{F} , the group algebra $\mathbb{F}G$ is left perfect if and only if G is finite. Subsequently, a generalization of these results to semigroup rings has been carried out by J. Okniński in [14]. On the other hand, although semiperfect group algebras have been also investigated in several papers (see e.g. [3, 6, 22, 24]), a full characterization is not available yet. The best partial result in this direction was obtained by J.M. Goursaud in [6] where, under the assumption that the group G is locally finite, it is proved that $\mathbb{F}G$ is semiperfect if and only if either $\text{char } \mathbb{F} = 0$ and G is finite or $\text{char } \mathbb{F} = p > 0$ and G has a normal p -group of finite index.

Date: October 21, 2016.

2010 Mathematics Subject Classification. 17B35, 16P70, 17B50.

The research of the second author was supported by NSERC of Canada under grant # RGPIN 418201.

In this paper, we consider these problems in the setting of enveloping algebras. For a restricted Lie algebra L over a field \mathbb{F} of characteristic $p > 0$, we denote by $u(L)$ the restricted enveloping algebra of L and by L' the derived subalgebra of L . Recall that a subset S of L is said to be p -nil if every element of S is p -nilpotent. We first provide some necessary and sufficient conditions on L such that $u(L)$ is semiperfect. We show that if $u(L)$ is semiperfect, then every element of L is p -algebraic and L does not contain any infinite-dimensional torus. Under the assumptions that \mathbb{F} is perfect and L' is p -nil we prove that for a locally finite-dimensional restricted Lie algebra L , $u(L)$ is semiperfect if and only if L contains a p -nil restricted ideal of finite codimension. We construct an infinite-dimensional abelian restricted Lie algebra L over an imperfect field K such that $u(L)$ is semiperfect and L has no nonzero p -nil restricted ideal. Hence, the perfectness assumption on the ground field is necessary. It turns out that the structure of semiperfect ordinary enveloping algebras $U(L)$ of arbitrary Lie algebras L is trivial and also quickly discussed.

In the last section, we prove that $u(L)$ is left (or right) perfect if and only if L is finite-dimensional, which represents the Lie-theoretic analogue of the aforementioned result of Renault and Woods.

2. SEMIPERFECTNESS

Throughout the paper, \mathbb{F} denotes a field. Of course, every field is a left and right perfect ring. To avoid any possible confusion, we recall that \mathbb{F} is called a perfect field if every finite extension of \mathbb{F} is separable.

It would be an interesting problem to characterize perfect and semiperfect Hopf algebras. Our first result, which we actually use later in the setting of restricted enveloping algebras, is a step towards this goal. Recall that an element Λ in a Hopf algebra H with comultiplication Δ and counit ϵ is called a left integral if $h\Lambda = \epsilon(h)\Lambda$, for every $h \in H$. Right integrals are defined analogously. Moreover, an element x of H is called *group-like* if $x \neq 0$ and $\Delta(x) = x \otimes x$ and *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$.

Theorem 2.1. *Suppose that a Hopf algebra H is semiperfect. Then H contains no infinite chain of semisimple Hopf subalgebras. Moreover, if H is generated as an algebra by its group-like and primitive elements, then all such elements are algebraic.*

Proof. For every subspace V of H , we write \bar{V} for the image of V in $H/\mathcal{J}(H)$. By contradiction, suppose that H contains an infinite chain of semisimple Hopf subalgebras

$$0 \subsetneq H_1 \subsetneq H_2 \subseteq \cdots \subsetneq H_n \subsetneq \cdots.$$

By Lemma 5.3.1 of [5], each Hopf algebra H_i is finite-dimensional. Moreover, by Theorem 2.2.1 and Corollary 2.2.4 of [13], left and right integrals of H_i coincide and form a 1-dimensional ideal of H_i which is not contained in $\ker \epsilon_i$, where ϵ_i denotes the counit of H_i . Therefore we can find $h_i \in \ker \epsilon_i$ such that $(1 + h_i)x = 0 = x(1 + h_i)$, for every $x \in \ker \epsilon_i$. Note that for every $i \leq j$ one has $(1 + h_j)(1 + h_i) = 1 + h_j = (1 + h_i)(1 + h_j)$. As a consequence, we have a chain

$$\overline{H(1 + h_1)} \supseteq \overline{H(1 + h_2)} \supseteq \cdots \supseteq \overline{H(1 + h_n)} \supseteq \cdots$$

of left ideals of \bar{H} . Now, since H is semiperfect, the algebra \bar{H} is left Artinian and so we must have

$$\overline{H(1 + h_n)} = \overline{H(1 + h_{n+1})}$$

for some n . Therefore, as the images of $1 + h_n$ and $1 + h_{n+1}$ in \bar{H} are commuting idempotents generating the same left ideal, they must coincide. Thus we have $h_{n+1} - h_n \in \mathcal{J}(H)$. But we have

$$(h_n - h_{n+1})^2 = (h_n + 1)^2 + (h_{n+1} + 1)^2 - 2(h_n + 1)(h_{n+1} + 1) = h_n - h_{n+1}.$$

We deduce that $h_{n+1} = h_n$, because $\mathcal{J}(H)$ is free of nonzero idempotents. In particular, $1 + h_{n+1} \in H_n$, a contradiction, yielding the first part of the theorem.

Now, if H is generated as an algebra by its group-like and primitive elements, then H is clearly pointed. Let x be a group-like or a primitive element of H and suppose, by contradiction, that x is transcendental. Set $x_0 = x$ and define recursively $x_{i+1} = x_i - x_i^2$ for every $n \geq 0$. Since $H/\mathcal{J}(H)$ is (right) Artinian, we deduce from Lemma 3.1 in [24] that there exists a positive integer m such that $1 - x_m$ has a right inverse y in H . Let B denote the Hopf subalgebra of H generated by x , so that we have $1 - x_m \in B$. Then B is either the polynomial algebra $\mathbb{F}[x]$ (when x is primitive) or the Laurent polynomial algebra $\mathbb{F}[[x]]$ (when x is group-like). In particular, B has no nonzero zero-divisors. Therefore, as H is pointed, it follows from the main theorem in [16] that H is a free left B -module, say $H = \bigoplus_{h \in S} Bh$. Note that by the proof of main theorem in [16], we can assume that $1 \in S$. Now we can deduce that y is contained in B . But the units of $\mathbb{F}[x]$ are non-zero elements of \mathbb{F} , and units of $\mathbb{F}[[x]]$ are of the form αx^k , where $0 \neq \alpha \in \mathbb{F}$ and k is an integer. This yields a contradiction, completing the proof. \square

As group algebras are obviously generated by group-like elements, we remark that Theorem 2.1 generalizes Theorem 3.2 in [24].

Let L be a restricted Lie algebra over a field of characteristic $p > 0$. For $S \subseteq L$ we denote by $\langle S \rangle_p$ the restricted subalgebra generated by S . An element x of L is said to be p -algebraic if $\dim \langle x \rangle_p < \infty$ and p -transcendental, otherwise. As ordinary and restricted enveloping algebras are generated by primitive elements, Theorem 2.1 yields the following:

Corollary 2.2. *Let L be restricted Lie algebra over a field of characteristic $p > 0$ such that $u(L)$ is semiperfect. Then L is p -algebraic and contains no infinite-dimensional torus.*

Proof. In view of Theorem 2.1, L is clearly p -algebraic. Now suppose, by contradiction, that L contains an infinite dimensional torus T . Clearly, T gives rise to an infinite chain

$$0 \subsetneq T_1 \subsetneq T_2 \subseteq \cdots \subsetneq T_n \subsetneq \cdots$$

of finite-dimensional tori of L . Then, by [7], we see that $u(T_i)$ is a semisimple Hopf subalgebra of $u(L)$ for every i , which contradicts Theorem 2.1. \square

Also, as the non-zero elements of an arbitrary Lie algebra L are transcendental in the ordinary enveloping algebra $U(L)$ of L , from Theorem 2.1 we deduce that $U(L)$ is semiperfect only in the trivial case:

Corollary 2.3. *Let L be a Lie algebra. Then the ordinary enveloping algebra $U(L)$ is semiperfect if and only if $L = 0$.*

Remark 2.4. As the trivial module of an ordinary enveloping algebra $U(L)$ is not projective unless $L = 0$, one can also prove Corollary 2.3 by using the fact that $U(L)$ is semiprimitive (see Theorem 8.3.14 in [19]) and so only projective $U(L)$ -modules have projective covers (see e.g. [19], Exercise 20 on page 326).

Let X be a basis of L and $u \in u(L)$. Then, by the Poincaré-Birkhoff-Witt (PBW) Theorem for restricted Lie algebras (see [21], Theorem 2.5.1), there exists a finite subset of X , denoted by $\text{Supp}(u)$, such that u can be written as a linear combination of PBW monomials in the elements of $\text{Supp}(u)$ only. We recall that a restricted Lie algebra is said to be *locally finite-dimensional* if all of its finitely generated restricted subalgebras are finite-dimensional.

Proposition 2.5. *Let L be a locally finite-dimensional restricted Lie algebra over a field of characteristic $p > 0$. Then $\mathcal{J}(u(L))$ is nil. In particular, $u(L)$ is semiperfect if and only if $u(L)/\mathcal{J}(u(L))$ is Artinian.*

Proof. Let $u \in \mathcal{J}(u(L))$. Since $\text{Supp}(u)$ is finite and L is locally finite-dimensional, we have that $H = \langle \text{Supp}(u) \rangle_p$ is finite-dimensional. Since $u \in H$, we deduce that $u(L)$ is algebraic and the assertion follows from [9, §1.10]. The second part is clear as idempotents can be always lifted modulo a nil ideal. \square

We need the following result for later use, however we were unable to find a reference for it and as such we include a proof for completeness.

Proposition 2.6. *Let R be a ring and N a nil ideal of R . If R/N is semiperfect, then so is R .*

Proof. Note that $\mathcal{J}(R/N) = \mathcal{J}(R)/N$. Hence, we have

$$(R/N)/\mathcal{J}(R/N) \cong (R/N)/(\mathcal{J}(R)/N) \cong R/\mathcal{J}(R).$$

Since R/N is Artinian, it follows that $R/\mathcal{J}(R)$ is Artinian.

Now, let $e \in R$ such that the image of e modulo $\mathcal{J}(R)$ is an idempotent in $R/\mathcal{J}(R)$. Since R/N is semiperfect, there exists $f \in R$ such that the image of f modulo N is an idempotent in R/N and $e - f \in \mathcal{J}(R)$. But N is nil, and it is well known that f can be lifted to an idempotent of R , that is, there exists $g \in R$ such that $f - g \in N \subseteq \mathcal{J}(R)$. Notice that $e - g \in \mathcal{J}(R)$ and we are done. \square

Theorem 2.7. *Let L be a locally finite-dimensional restricted Lie algebra over a field of characteristic $p > 0$. If L has a p -nil restricted ideal of finite codimension, then $u(L)$ is semiperfect.*

Proof. Let P be a p -nil restricted ideal of finite codimension. We first prove that the ideal $I = Pu(L)$ is nil. To do so, let $u \in I$. Then $u = \sum_i^n p_i u_i$, where each $p_i \in P$ and each $u_i \in u(L)$. Let \mathcal{B} be a basis of L containing a basis of P . Note that there exists a finite subset S of \mathcal{B} such that all the elements p_i, u_i are in $u(H)$, where H is the restricted subalgebra of L generated by S . Now, let Q be the restricted ideal of H generated by all the p_i 's. Then $Q \subseteq P \cap H$, and so we see that Q is a finite-dimensional p -nilpotent restricted ideal of H . Note that $Qu(H)$ is associative nilpotent. Since $u \in Qu(H)$, u is nilpotent and as such I is nil. Now, note that

$$u(L/P) \cong u(L)/I.$$

Since L/P is finite-dimensional, clearly $u(L/P)$ is semiperfect. Hence, $u(L)/I$ is semiperfect and it follows from Proposition 2.6 that $u(L)$ is semiperfect. \square

We remark that an argument similar to the one used in Theorem 2.7 yields an alternative and shorter proof of the analogous result for group rings obtained in [24].

For a group algebra $\mathbb{F}G$ of a locally finite group G over a field \mathbb{F} of characteristic $p > 0$, it is shown by J.M. Goursaud (see [6], Théorème 8) that $\mathbb{F}G$ is semiperfect if and only if $[G : \mathcal{O}_p(G)] < \infty$, where $\mathcal{O}_p(G)$ is the largest normal p -subgroup of G . In analogy, we let $\mathcal{O}_p(L)$ be the sum of all p -nil restricted ideals of L . Since the sum of every two p -nil restricted ideals of L is again p -nil, it follows that $\mathcal{O}_p(L)$ is also a p -nil restricted ideal. Clearly, if L is finite-dimensional then $\mathcal{O}_p(L)$ is just $\text{rad}_p(L)$, the p -radical of L . Therefore one might expect that the analogous result holds for restricted Lie algebras, that is, $u(L)$ is semiperfect if and only if $\dim L/\mathcal{O}_p(L) < \infty$. In Proposition 2.8 we provide a partial converse of Theorem 2.7 and, on the other hand, in Example 2.11 we present an infinite-dimensional abelian restricted Lie algebra L over an imperfect field such that $u(L)$ is semiperfect and yet $\mathcal{O}_p(L) = 0$. Thus, even in the abelian case, the Lie-theoretic analogue of Goursaud's Theorem does not hold in general.

Proposition 2.8. *Let L be a locally finite-dimensional restricted Lie algebra over a perfect field \mathbb{F} of characteristic $p > 0$. If $u(L)$ is semiperfect and L' is p -nil, then $\mathcal{O}_p(L)$ has finite codimension in L .*

Proof. First we show that the associative ideal $I = \mathcal{O}_p(L)u(L)$ is nil. Let $u \in I$. Then $u = \sum x_i u_i$, where $x_i \in \mathcal{O}_p(L)$ and $u_i \in u(L)$. Now let H be the restricted subalgebra generated by all the x_i 's and all the $\text{Supp}(u_i)$'s. Note that H is finite-dimensional. Let N be the restricted ideal of H generated by the x_i 's. Since $N \subseteq \mathcal{O}_p(L)$, we deduce that N is finite-dimensional and p -nilpotent. It follows that $Nu(H)$ is nilpotent. In particular, as $u \in Nu(H)$, we deduce that u is nilpotent. Therefore, I is a nil ideal and as such $I \subseteq \mathcal{J}(u(L))$. We now claim that $I = \mathcal{J}(u(L))$. Let $\mathcal{L} = L/\mathcal{O}_p(L)$. Since $L' \subseteq \mathcal{O}_p(L)$, \mathcal{L} is free of nonzero p -nilpotent elements. We claim that $u(\mathcal{L}) \cong u(L)/I$ is reduced. Let u be a nilpotent element of $u(\mathcal{L})$. There exist elements x_1, \dots, x_n in L that are linearly independent modulo $\mathcal{O}_p(L)$ such that

$$u = \sum \alpha x_1^{i_1} \cdots x_n^{i_n} \quad \text{modulo } I.$$

Thus, $u^{p^r} = 0$ for some positive integer r . Hence, by the PBW Theorem, the elements $x_1^{[p]^r}, \dots, x_n^{[p]^r}$ are linearly dependent modulo $\mathcal{O}_p(L)$. Let $\beta_1, \beta_2, \dots, \beta_n \in \mathbb{F}$, not all zero, such that

$$\sum \beta_k x_k^{[p]^r} \in \mathcal{O}_p(L).$$

Since \mathbb{F} is a perfect field, we get

$$\left(\sum \beta_k^{\frac{1}{p^r}} x_k \right)^{[p]^r} \in \mathcal{O}_p(L).$$

But \mathcal{L} has no nonzero p -nilpotent elements. We deduce that $\sum \beta_k^{\frac{1}{p^r}} x_k \in \mathcal{O}_p(L)$, which contradicts the fact that x_1, \dots, x_n are linearly independent modulo $\mathcal{O}_p(L)$. Hence, $u(\mathcal{L})$ is reduced. On the other hand, by Proposition 2.5, $\mathcal{J}(u(\mathcal{L}))$ is nil. We deduce that $\mathcal{J}(u(\mathcal{L})) = 0$. Thus, $\mathcal{J}(u(L)) \subseteq I$, as claimed. At this stage, as $u(\mathcal{L}) \cong u(L)/I = u(L)/\mathcal{J}(u(L))$ is Artinian, we deduce from [12] that \mathcal{L} is indeed finite-dimensional. Therefore $\mathcal{O}_p(L)$ has finite codimension in L , as required. \square

As an immediate consequence of Theorem 2.7 and Proposition 2.8 we have the following characterization of semiperfect commutative restricted enveloping algebras over perfect fields:

Corollary 2.9. *Let L be an abelian restricted Lie algebra over a perfect field of characteristic $p > 0$. Then $u(L)$ is semiperfect if and only if $\mathcal{O}_p(L)$ has finite codimension in L .*

It is worth mentioning that, in general, the properties of perfectness or semiperfectness of an algebra are not preserved under extensions of the ground field. For instance, consider the following:

Example 2.10. Let \mathbb{F} be an algebraically closed field and let $E = \mathbb{F}(t, t^{\frac{1}{2}}, t^{\frac{1}{4}}, \dots)$ be the extension of the field $\mathbb{F}(t)$ of rational functions in the variable t obtained by adjoining all the 2-power roots of t . Consider the automorphism $\alpha : E \rightarrow E$, $f(t) \mapsto f(t^2)$ and form the division algebra $D = E((x; \alpha))$ of skew Laurent series (for details, see e.g., Section 2.3 of [4]). Obviously, D is left perfect. Now, let $K = \mathbb{F}((x; \alpha))$. Then K is a maximal subfield of D and, clearly, the algebra D is not algebraic over K . Moreover, as \mathbb{F} is algebraically closed, K is a transcendental field extension of \mathbb{F} . Therefore, Theorem 2.4 of [10] assures that $\mathcal{J}(D \otimes_{\mathbb{F}} K) = 0$. Finally, by Exercise 6 in Section IX.6 of [8], we see that $D \otimes_{\mathbb{F}} K$ is not left Artinian and, indeed, not semiperfect.

We remark that the analogue of the aforementioned result of Goursaud for restricted Lie algebras fails without the perfectness assumption on the field. We show as follows that the restriction on the ground field is required. For simplicity, we construct our example over a field \mathbb{F} of characteristic 2 but this example could be extended to any positive characteristic. In the following we denote by K the field of rational functions in infinitely many indeterminates t_1, t_2, \dots over \mathbb{F} , that is, $K = \mathbb{F}(t_1, t_2, \dots)$.

Theorem 2.11. *There exists an infinite-dimensional abelian restricted Lie algebra L over K such that $u(L)$ is semiperfect and $\mathcal{O}_p(L) = 0$.*

Proof. Let L be the abelian restricted Lie algebra over K with basis x, y_1, y_2, \dots and power mapping defined by $x^{[2]} = x, y_i^{[2]} = t_i x$ for $i \geq 1$.

We have that $\mathcal{O}_p(L) = 0$. Indeed, let $z \in L$ such that $z^{[2]} = 0$. Then $z = \alpha x + \sum \beta_i y_i$ for some $\alpha, \beta_i \in K$. We have

$$z^{[2]} = (\alpha^2 + \sum \beta_i^2 t_i) x = 0.$$

Hence, $\alpha^2 + \sum \beta_i^2 t_i = 0$. It is easy to deduce that $\alpha = \beta_i = 0$ for all i . Hence, $z = 0$ and $\mathcal{O}_p(L) = 0$. We denote by \bar{u} the image of an element u modulo $\mathcal{J}(u(L))$. Note that L is locally finite-dimensional. Hence, to show that $u(L)$ is semiperfect, by the second part of Proposition 2.5 it is enough to prove that $\bar{R} = u(L)/\mathcal{J}(u(L))$ is Artinian. For this, we will first show that $u(L)/\mathcal{J}(u(L))$ satisfies the descending chain condition on principal ideals by proceeding in the following steps.

Claim 1: *Let $u \in u(L)$. Then \bar{u} is invertible in $u(L)/\mathcal{J}(u(L))$ if and only if u is invertible in $u(L)$. Suppose $\bar{u}\bar{v} = \bar{1}$, for some $v \in u(L)$. Then $1 - uv$ is nilpotent. Thus, uv is invertible which implies that u is invertible.*

Claim 2: Let $u = a + bx \in u(L)$, where $a, b \in K$. Then u is invertible if and only if $a \neq 0$ and $a \neq b$. Suppose that $a \neq 0$ and $a \neq b$. Then we have

$$(a + bx)(a^{-1} + b(a^2 + ab)^{-1}x) = 1.$$

The converse follows from the fact that $x(x + 1) = 0$.

Claim 3: Let $0 \neq d \in K^{2^r}(t_1^{2^{(r-1)}}, t_2^{2^{(r-1)}}, \dots)$, where $r \geq 1$. Then there exists $w \in u(L)$ such that $w^{2^r} = d^{-1}x$. Suppose that

$$d = \beta^{2^r} + \sum_{n \geq 1} \gamma_{i_1, \dots, i_n}^{2^r} (t_{i_1} \cdots t_{i_n})^{2^{(r-1)}}$$

for some β and $\gamma_{i_1, \dots, i_n} \in K$. Put

$$w = \frac{\beta x + \sum_{n \geq 1} \gamma_{i_1, \dots, i_n} y_{i_1} \cdots y_{i_n}}{\beta^2 + \sum_{n \geq 1} \gamma_{i_1, \dots, i_n}^2 t_{i_1} \cdots t_{i_n}}.$$

We observe that $w^{2^r} = d^{-1}x$.

Claim 4: For any chain of principal ideals $0 \subsetneq (\bar{u}_2) \subseteq (\bar{u}_1) \subsetneq \bar{R}$, we have that $(\bar{u}_2) = (\bar{u}_1)$. Let $v \in u(L)$ such that $\bar{u}_2 = \bar{u}_1 \bar{v}$. We may assume that \bar{v} is not invertible. We show that there exists $w \in u(L)$ such that $u_1 - u_1 v w$ is nilpotent. Note that $(y_i(x - 1))^2 = 0$, for every y_i . It follows that $\omega(L)(x - 1) \subseteq \mathcal{J}(u(L))$. In particular, by the PBW Theorem, every element $u \in u(L)$ can be written in the form

$$u = \alpha + \beta x + \sum \gamma_{i_1, \dots, i_n} y_{i_1} \cdots y_{i_n} \quad \text{modulo } \mathcal{J}(u(L)).$$

Thus, for large r , we have

$$u^{2^r} = \alpha^{2^r} + \beta^{2^r} x + \sum \gamma_{i_1, \dots, i_n}^{2^r} (t_{i_1} \cdots t_{i_n})^{2^{(r-1)}} x.$$

We deduce that $u^{2^r} = a + bx$, where $a \in K^{2^r}$ and $b \in K^{2^r}(t_1^{2^{(r-1)}}, t_2^{2^{(r-1)}}, \dots)$.

Let $u_1^{2^r} = a + bx$ and $v^{2^r} = c + dx$, where $a, c \in K^{2^r}$ and $b, d \in K^{2^r}(t_1^{2^{(r-1)}}, t_2^{2^{(r-1)}}, \dots)$. Since \bar{u}_1 is not invertible, it follows from Claim 2 that either $a = 0$ or $a = b$. Similarly, since \bar{v} is not invertible, either $c = 0$ or $c = d$.

Suppose first that $a = 0$. Then, if $c = d$, we get $\bar{u}_2^{2^r} = \bar{u}_1^{2^r} \bar{v}^{2^r} = bcx(1 + x) = 0$. Hence, $u_2 \in \mathcal{J}(u(L))$ and so $\bar{u}_2 = 0$ which is a contradiction to the assumptions. We deduce that, if $a = 0$, then $c = 0$ and $c \neq d$. Now, by Claim 3, there exists $w \in u(L)$ such that $w^{2^r} = d^{-1}x$. It is easy to see that $u_1^{2^r} - (u_1 v w)^{2^r} = 0$. Now, assume that $a \neq 0$ and so $a = b$. In a similar way as above, we can see that $c \neq 0$ and $c = d$. Since $c \in K^{2^r}$, there exists $w \in u(L)$ such that $w^{2^r} = c^{-1}$. It is clear that $u_1^{2^r} - (u_1 v w)^{2^r} = 0$. This completes the proof of Claim 4.

It now follows from Claim 4 that $u(L)/\mathcal{J}(u(L))$ satisfies the descending chain condition on principal ideals. Since $u(L)/\mathcal{J}(u(L))$ is semiprime, we deduce from Theorem 10.24 in Chapter 4 of [11] that $u(L)/\mathcal{J}(u(L))$ is Artinian, as required. \square

3. PERFECTNESS

In this section, we characterize left perfect restricted enveloping algebras $u(L)$. As the antipode of any cocommutative Hopf algebra is an involution, left and right perfectness are, in fact, equivalent conditions for $u(L)$.

Lemma 3.1. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $u(L)$ is left perfect, then L is finitely generated.*

Proof. Let $R = u(L)$. Suppose, by contradiction, that L is not finitely generated and let $x_1, x_2, \dots \in L$ so that $x_{k+1} \notin \langle x_1, \dots, x_k \rangle_p$. Now consider the principal right ideals

$$x_1 R \supseteq x_1 x_2 R \supseteq \dots \supseteq x_1 x_2 \dots x_k R \supseteq \dots$$

Since, according to Bass' Theorem (see [1], Theorem P), R satisfies the descending chain condition on principal right ideals, there exists an integer i such that

$$x_1 \dots x_i R = x_1 \dots x_i x_{i+1} R.$$

Therefore one has

$$x_1 \dots x_i - x_1 \dots x_i x_{i+1} v = 0, \tag{3.1}$$

for some $v \in u(L)$. Let H be the restricted subalgebra of L generated by x_1, \dots, x_i . We extend x_1, \dots, x_i to a basis X of H and extend X to a basis $X \cup Y$ of L . Since $x_{i+1} \notin H$, we can assume that $x_{i+1} \in Y$. It is well known that $u(L)$ is a free left $u(H)$ -module, that is,

$$u(L) = \oplus_i u(H) u_i,$$

where the u_i 's are distinct PBW monomials in the elements of Y . Now, we observe that Equation (3.1) is not possible because some monomials in the PBW representation of $x_{i+1} v$ involve x_{i+1} and this contradicts $u(L)$ being a free $u(H)$ -module. Hence, L must be finitely generated. \square

Lemma 3.2. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $u(L)$ is left perfect, then so is $u(H)$ for every restricted subalgebra H of L .*

Proof. Let Y be a basis for a complementary vector subspace of H in L . Since $u(L)$ is a free left $u(H)$ -module, we have

$$u(L) = \oplus_k u(H) u_k, \tag{3.2}$$

where the elements u_k are distinct PBW monomials in the elements of Y . Let $h \in u(H)$ and let $I = hu(H)$ be the right ideal of $u(H)$ generated by h . Then (3.2) implies that

$$J = \oplus_k I u_k$$

is a principal right ideal of $u(L)$. As a consequence, every descending chain $\{I_k\}_k$ of principal right ideals of $u(H)$ gives rise to a descending chain of principal right ideals of $u(L)$ which must stabilize, as $u(L)$ is left perfect. Therefore, by (3.2), the chain $\{I_k\}_k$ stabilizes, which finishes the proof. \square

For a restricted Lie algebra L , we define

$$\Delta(L) = \{x \in L \mid \dim [L, x] < \infty\}.$$

Then $\Delta(L)$ is clearly a restricted ideal of L which is the Lie algebra analogue of the FC-center of a group.

Proposition 3.3. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. If $L = \Delta(L)$ and every element of L is p -algebraic, then L is locally finite-dimensional.*

Proof. Let $x_1, \dots, x_n \in L$ and let H be the (ordinary) subalgebra of L generated by these elements. By Lemma 1.3 of [2] we have that $\dim[H, H] < \infty$. Since H is spanned as a vector subspace by x_1, \dots, x_n and their Lie commutators, we see that H is indeed finite-dimensional. Since every element of H is p -algebraic, it follows from Proposition 1.3(1) in Chapter 2 of [21] that $\langle x_1, \dots, x_n \rangle_p$ is finite-dimensional, as required. \square

The following result is an immediate consequence of Corollary 6.4 in [2]:

Lemma 3.4. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. Then $u(L)$ is semiprime if and only if $u(\Delta(L))$ is semiprime.*

We can now prove the main result of this section:

Theorem 3.5. *Let L be a restricted Lie algebra over a field of characteristic $p > 0$. Then $u(L)$ is left perfect if and only if L is finite-dimensional.*

Proof. One implication is clear. Suppose now that $u(L)$ is left perfect. By Lemma 3.2, $u(\Delta(L))$ is also left perfect and so, by Lemma 3.1, $\Delta(L)$ is finitely generated. Note that, by Corollary 2.2, every element of $\Delta(L)$ is p -algebraic. Hence, by Proposition 3.3, we conclude that $\Delta(L)$ is finite-dimensional. Thus, it is enough to prove that $L/\Delta(L)$ is finite-dimensional. Note that, by Corollary 24.19 of [11], the ring $u(L/\Delta(L)) \cong u(L)/\Delta(L)u(L)$ is left perfect, as well. Hence, we can replace L by $L/\Delta(L)$. By [18, Lemma 2.7(i)], we have that $\Delta(L/\Delta(L)) = 0$. Therefore we can assume that $\Delta(L) = 0$. Now, Lemma 3.4 entails that $u(L)$ is semiprime and, by Bass' Theorem (see [1, Theorem P]), $u(L)$ satisfies the descending chain condition on principal right ideals. But then, by Theorem 10.24 in [11], $u(L)$ is right Artinian. Since, by [12], right Artinian Hopf algebras are finite-dimensional, we conclude that L is finite-dimensional. This completes the proof. \square

Some immediate consequences of Theorem 3.5 are now in order. As the category of (left) $u(L)$ -modules is equivalent to the category of restricted (left) L -modules, a combination of Theorem 3.5 and Theorem 24.25 in [11] yields the following:

Corollary 3.6. *Let L be a restricted Lie algebra. The following conditions are equivalent:*

- (1) *every restricted flat L -module is projective;*
- (2) *L is finite-dimensional.*

Let R be a ring and $\theta : G \rightarrow \text{Aut}(R)$ be a group homomorphism. Then J.K. Park in [15] proved that the smash product $R \# \mathbb{F}G$ is left perfect if and only if R is left perfect and G is finite. In particular, in view of Theorem 3.5, if $R = u(L)$ for a restricted Lie algebra L , we deduce the following:

Corollary 3.7. *Let $A = u(L) \# \mathbb{F}G$. Then A is left perfect if and only if A is finite-dimensional.*

Note also that Theorem 3.5 generalizes Corollary 6.6 in [2]. Finally, the celebrated structure theorem of Cartier-Kostant-Milnor-Moore (see e.g. [13, §5.6]) implies that every cocommutative Hopf algebra over an algebraically closed field of characteristic zero can be presented as a smash product of a group algebra and an enveloping algebra. So, Corollary 2.3 along with Park's result in [15] yields:

Corollary 3.8. *Let H be a cocommutative Hopf algebra over an algebraically closed field \mathbb{F} of characteristic zero. Then H is left perfect if and only if H is the group algebra of a finite group G over \mathbb{F} .*

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